# Lower Bounds from Cyclotomic Divisors of Mask Polynomials

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# Tilings of the integers



Let $A \subset \mathbb{Z}$ be a finite set.
We say that $A$ tiles $\mathbb{Z}$ (by translation) if there is a $T \subset \mathbb{Z}$ such that $\forall n \in \mathbb{Z}$ can be uniquely expressed as a sum $a + t = n$, with $a \in A$ and $t \in T$.

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For such a $B$ we have $|A||B| = M$ and $A \oplus B = \mathbb{Z}_M$. All results on $\mathbb{Z}_M$ can be translated back to the integer setting. Thus, from now on, we will work on $\mathbb{Z}_M$.

For any $s|M$, we have $\Phi_s \mid (X^M - 1)$, so that $\Phi_s|A$ if and only if $\Phi_s \mid (A \bmod M)$.

Note that $A \bmod M$ need not be a set hence we introduce the multiset notation.

- $\mathcal{M}(\mathbb{Z}_M)$ denote the set of all multisets in $\mathbb{Z}_M$ with weights in $\mathbb{Z}$ (so that both positive and negative weights are allowed)
- $\mathcal{M}^+(\mathbb{Z}_M)$ if we only allow positive weights.

For $a \in \mathbb{Z}_M$, let $w_A(a)$ denote the weight of $a$ in $A$.

The mask polynomial of the multiset $A$ by

$$A(X) = \sum_{a \in \mathbb{Z}_M} w_A(a) X^a.$$

In particular, $A \in \mathcal{M}(\mathbb{Z}_M)$ is a set if and only if $w_A(x) \in \{0, 1\}$ for all $x \in \mathbb{Z}_M$.

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Given a set $S = \{s_1, s_2, \ldots, s_k\}$ of divisors of $M$.

Our goal would be to decide whether there exists a set / tile $A$ satisfying $\Phi_{s_j}(X) \mid A(X)$ $(1 \leq j \leq k)$, and

$$|A| = A(1) = \prod_{p_i^{m_i} \in S} \Phi_{p_i^{m_i}}(1).$$

# Coven-Meyerowitz theorem



Theorem (Coven and Meyerowitz, 1999)

*Let $A$ be a finite set of integers and*

$$S_A^* := \{p^\alpha \,:\, p^\alpha \text{ is a prime power and } \Phi_{p^\alpha}(X) | A(X)\}.$$

*Consider the following two conditions*
*(T1)* $|A| = A(1) = \prod_{s \in S_A^*} \Phi_s(1)$,
*(T2) if* $s_1, \ldots, s_k \in S_A^*$ *are powers of different primes, then* $\Phi_{s_1 \ldots s_k}(X) | A(X)$.

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*Then*

- *If $A$ satisfies both (T1) and (T2) then $A$ tiles $\mathbb{Z}$.*
- *If $A$ tiles $\mathbb{Z}$ then it must satisfy (T1).*
- *If $A$ tiles $\mathbb{Z}$, and $|A|$ has at most two prime factors, then it satisfies (T2).*

# Conjecture and generalization



**Conjecture (Coven-Meyerowitz conjecture)**

*A set tiles the integers if and only if it satisfies (T1) and (T2).*

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It is still open. But it is known to be true in some special cases.

**Theorem (Łaba-Londner, 2025)**

*The Coven-Meyerowitz conjecture is true in $\mathbb{Z}_M$ if any of the following conditions holds.*

- $M \mid p_1^m p_2^n \prod_{i=3}^{L} p_i$,
- $M \mid p_1^2 p_2^2 p_3^2 \prod_{i=4}^{L} p_i$,

*where $p_1, p_2, p_3, p_i$'s are distinct primes.*

# A strategy to verify CM-conjecture



One possible avenue of approach is to consider (T1) as an upper bound on the size of $A$, and ask whether a set obeying this bound may have additional cyclotomic divisors that would allow a failure of (T2) for its tiling complement.

# A strategy to verify CM-conjecture



One possible avenue of approach is to consider (T1) as an upper bound on the size of $A$, and ask whether a set obeying this bound may have additional cyclotomic divisors that would allow a failure of (T2) for its tiling complement. The details are as follows.

**Definition**

Let $A \subset \mathbb{Z}_M$, and let $\Phi_s(X) \mid A(X)$ for some $s \mid M$. We say that $\Phi_s$ is an *unsupported divisor of $A$* if:

(i) for every prime $p$ such that $p \mid s$, we have $p \mid |A|$,

(ii) for every prime power $p^\alpha$ such that $p^\alpha \parallel s$, we have $\Phi_{p^\alpha} \nmid A$.

# Questions



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## Question (B)

*If $A \subset \mathbb{Z}_M$ satisfies (T1) and (T2), may it have unsupported divisors?*

## Proposition

*Let $A \oplus B = \mathbb{Z}_M$ such that each prime factor of $M$ divides both $|A|$ and $|B|$.*

(i) *If the answer to Question (A) is negative for this value of $M$, then both sets $A$ and $B$ satisfy (T2).*

(ii) *If the answer to Question (B) is negative for this value of $M$, then, if (T2) holds for $A$, it must also hold for $B$.*

**Theorem**

*There exists* $M = p^n q^m$ *and a nonempty set* $A \subset \mathbb{Z}_M$ *satisfying (T1), and* $A(X)$ *has at least one unsupported cyclotomic divisor.*

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However, in the 'two-prime-divisor' case we can prove the following.

**Theorem**

*Let $M = p^n q^m$. Assume that a nonempty set $A \subset \mathbb{Z}_M$ satisfies (T1) and (T2). Then $A(X)$ cannot have unsupported cyclotomic divisors.*

# Lower bound for the size of sets with given cyclotomic divisors



Let $S = \{s_1, s_2, \ldots, s_k\}$ be the divisors of $M$, and $A$ be a nonempty set in $\mathbb{Z}_M$ such that $\Phi_{s_j}(X) \mid A(X)$ $(1 \leq j \leq k)$.

**Question**

*What is the minimal size of A?*

# Lower bound for the size of sets with given cyclotomic divisors



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**Question**

*What is the minimal size of A? i.e,*

$$\mathrm{MIN}(S) := \min\{|A| : \ A \neq \emptyset \textit{ and } \Phi_s(X) \mid A(X) \textit{ for all } s \in S\}?$$

Motivation:

**Proposition (Lam and Leung)**

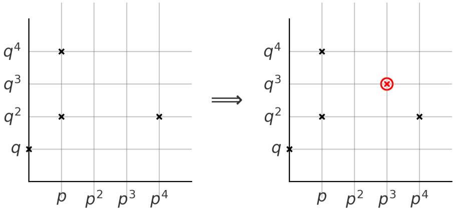
*If $\Phi_s(X)|A(X)$ for some $1 < s \in \mathbb{N}$, then*

$$|A| \geq \min\{p : p \mid s, p \text{ is prime}\}.$$

# Illustration



Take some cyclomatic divisors of $M = p^4q^4$ and we add an extra divisor as below.




$S = \{q, pq^2, pq^4, p^4q^2\}$

What is the minimum size of $A$ s.t. $\Phi_q\Phi_{pq^2}\Phi_{pq^4}\Phi_{p^4q^2} \mid A(X)$?

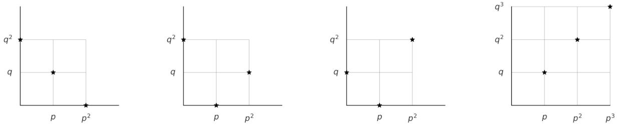
Is it smaller than the min. size of $A$ if $\Phi_{p^3q^3} \mid A(X)$ is added?

$$\text{Is MIN}(S) < \text{MIN}(S \cup \{p^3q^3\})?$$

# Examples



If $|S| = |\{s_1, s_2\}| = 2$ such that $s_i \nmid s_j$ $(i \neq j \in \{1,2\})$, then it is easy to show that $\mathrm{MIN}(S) \geq (\min(p,q))^2$.




In the first three cases we get that $\mathrm{MIN}(S) \geq pq \min(p,q)$.

In the last case $\Phi_{pq}\Phi_{p^2q^2}\Phi_{p^3q^3} \mid A(X)$ holds and we get that $\mathrm{MIN}(S) \geq (\min(p,q))^3$.

# Generalizations I.



**Proposition**

*Let* $A \in \mathcal{M}^+(\mathbb{Z}_M)$ *with* $M = p^{n_1} q^{n_2}$. *Assume that* $\Phi_{p^{m_1}} \cdots \Phi_{p^{m_r}} \Phi_{p^\alpha q^\beta} \Phi_{q^\gamma} | A$

- *for some* $1 \leq \alpha < m_1 < \cdots < m_r \leq n_1$ *and* $1 \leq \beta \neq \gamma \leq n_2$,
- *for some* $1 \leq m_1 < \cdots < m_r < \alpha \leq n_1$ *and* $1 \leq \gamma < \beta \leq n_2$.

*Then* $|A| \geq p^r q \min(p, q)$. *Hence* $\mathrm{MIN}(S) \geq p^r q \min(p, q)$.

# Generalization II.- Diagonal case



Let $s = \prod_{i=1}^{L} p_i^{\beta_i}$, then $D(s) = \frac{s}{\prod_{i=1}^{L} p_i}$

**Theorem**

*Let* $M = \prod_{i=1}^{L} p_i^{n_i}$. *Assume that* $S = \{s_1, \ldots, s_m\}$ *satisfies* $s_j \mid M$ *and*

$$s_j \mid D(s_{j+1}) \text{ for } j = 1, \ldots, m-1. \tag{1}$$

*Then* $\mathrm{MIN}(S) \geq \prod_{j=1}^{m} \min_{i:p_i|s_j} p_i$

# Examples without size increase



**Proposition**

*Let* $M = p^n q^m$ *with* $n \geq 9$ *and* $m \geq 6$*, and let* $p = 2, q = 3$*. Then there exists a set* $A \subset \mathbb{Z}_M$ *such that*

$$\Phi_{p^n}\Phi_{p^{n-1}}\Phi_{p^{n-2}}\Phi_{q^m}\Phi_{q^{m-1}}\Phi_{q^{m-2}}\Phi_{pq} \mid A$$

*and* $|A| = p^3q^3 = 216$*.*

**Proposition**

*Let* $M = p^4q^4$*,* $p = 2, q = 3$*. There exists a set* $A \subset \mathbb{Z}_M$ *such that*

$$\Phi_p\Phi_{p^2}\Phi_{p^3}\Phi_q\Phi_{q^2}\Phi_M \mid A$$

*and* $|A| = p^3q^2 = 72$*.*

# Fibers, fibered sets



Let $N \mid M$, and let $p_i$ be a prime such that $p_i \mid N$. We define

$$F_i^N(X) = \Phi_{p_i}(X^{N/p_i}) = 1 + X^{N/p_i} + \cdots + X^{(p_i-1)N/p_i},$$

which the mask polynomial of the set
$F_i^N = \{0, N/p_i, \ldots, (p_i - 1)N/p_i\} \mod N$.

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A *$p_i$-fiber on scale $N$* is a translate of $F_i^N$.

$A \subset \mathbb{Z}_M$ is *fibered* on scale $N$ if there exists a prime $p_i | N$ and there exists a polynomial $Q(X)$ with nonnegative integer coefficients such that

$$Q(X)F_i^N(X) \equiv A(X) \mod x^N - 1.$$

# de Bruin-Rédei-Schoenberg theorem



**Theorem**

*Let $A \in \mathcal{M}(\mathbb{Z}_M)$. Then the following are equivalent:*

(i) $\Phi_N(X) | A(X)$,

(ii) *A mod N is a linear combination of N-fibers, so that*

$$A(X) = \sum_{i : p_i | N} P_i(X) F_i^N(X) \mod X^N - 1,$$

*where $P_i(X)$ have integer coefficients.*

**Proposition (de Bruin, Lam-Leung)**

*Let $A \in \mathcal{M}^+(\mathbb{Z}_M)$. Assume that $\Phi_N | A$, where $N$ has two distinct prime factors $p_1, p_2$. Then*

$$A(X) = P_1(X) F_1^N(X) + P_2(X) F_2^N(X) \mod X^N - 1,$$

*where $P_1, P_2$ are polynomials with nonnegative coefficients.*

# Long fibers



**Definition**

Let $M = \prod_{i=1}^{K} p_i^{n_i}$, and let $1 \leq \alpha \leq n_i$. We say that a set $F \subset \mathbb{Z}_M$ is a *$p_i^\alpha$-fiber on scale $M$* if $F \equiv x * F_{i,\alpha} \bmod M$ for some $x \in \mathbb{Z}_M$, where

$$F_{i,\alpha}(X) := \prod_{\nu=1}^{\alpha} \Phi_{p_i}(X^{M/p_i^\nu}) \equiv \frac{X^M - 1}{X^{M/p_i^\alpha} - 1}.$$

We refer to $p_i^\alpha$-fibers with $\alpha > 1$ as *long fibers* in the $i$ direction.

$$F_{i,\alpha}(X) = 1 + X^{M/p_i^\alpha} + X^{2M/p_i^\alpha} + \cdots + X^{(p_i^\alpha - 1)M/p_i^\alpha}.$$

# Generalization of de Bruin-Rédei-Schoenberg theorem



### Proposition

*Long fiber decomposition Let* $M = \prod_{i=1}^{K} p_i^{n_i}$*, and let* $N|M$ *satisfy* $N = \prod_{i=1}^{K} p_i^{n_i-\alpha_i+1}$ *with* $1 \le \alpha_i \le n_i$*. Let* $A \in \mathcal{M}(\mathbb{Z}_M)$*, and assume that* $\Phi_L(X) \mid A(X)$ *for each* $N \mid L \mid M$*. Then, there exist polynomials* $P_i(X) \in \mathbb{Z}[X]$ *such that*

$$A(X) = P_1(X)F_{1,\alpha_1}(X) + \cdots + P_K(X)F_{K,\alpha_K}(X) \mod X^M - 1.$$

*Moreover, if* $A \in \mathcal{M}^+(\mathbb{Z}_M)$ *and* $K = 2$*, then we may assume that the polynomials* $P_1(X)$ *and* $P_2(X)$ *each have non-negative coefficients.*

# Truncation I.



The truncation procedure allows us to reduce proving lower bounds on MIN($S$) to proving similar bounds with $S$ replaced by a simpler set.

In order to discuss the statement we need the following definition.

**Definition**

Let $S$ be the subset of the div. of $M$ and $1 \leq i \leq K$ the number of prim div. of $M$, we define

$$\mathrm{EXP}_i(S) := \{\alpha \geq 1 : \ \exists\, s \in S \text{ with } p_i^{\alpha} \mid\mid s\}, \quad E_i := \#\mathrm{EXP}_i(S).$$

It will be useful to arrange the sets $\mathrm{EXP}_i(S)$ in increasing order:

$$\mathrm{EXP}_i(S) := \{\alpha_{i,1}, \cdots, \alpha_{i,E_i}\}, \quad 1 \leq \alpha_{i,1} < \cdots < \alpha_{i,E_i}.$$

# Truncation II.



**Proposition (Truncations)**

*Let $S$ be a subset of the divisors of $M$, and let $A \in \mathcal{M}(\mathbb{Z}_M)$ satisfy $\Phi_s | A$ for all $s \in S$. Define $M' := p_1^{E_1} \cdots p_K^{E_K}$. Then, there exists a multiset $A' \in \mathcal{M}(\mathbb{Z}_{M'})$ satisfying*

(i) $A'(1) = A(1)$.

(ii) *For every $N = p_1^{\alpha_{1,\ell_1}} \cdots p_K^{\alpha_{K,\ell_K}} \in S$, we have $\Phi_{N'}(X) \mid A'(X)$, where $N' := p_1^{\ell_1} \cdots p_K^{\ell_K} \mid M'$.*

*Furthermore, if $A \in \mathcal{M}^+(\mathbb{Z}_M)$, then $A' \in \mathcal{M}^+(\mathbb{Z}_{M'})$.*

# Truncation- An example



Suppose that $\Phi_s | A$ for all $s \in S$, where

$$S := \{p_1^3, p_2^2, p_2^4, , p_1^3 p_2^4, p_1^{10}, p_2^{10}, p_1^{10} p_2^{10}\}.$$

Then, $\mathrm{EXP}_1(S) = \{3, 10\}$ and $\mathrm{EXP}_2(S) = \{2, 4, 10\}$ so that $M' := p_1^{E_1} p_2^{E_2} = p_1^2 p_2^3$.

Then the truncation procedure furnishes a multiset $A' \in \mathcal{M}(\mathbb{Z}_{p_1^2 p_2^3})$ such that $A'(1) = A(1)$ and $\Phi_s | A'$ for all $s \in S' := \{p_1, p_2, p_2^2, p_1 p_2^2, p_1^2, p_2^3 p_1^2 p_2^3\}$.

The exponent sets associated to $A'$ are $\{1, 2\}$ and $\{1, 2, 3\}$, with no gaps.

# The cyclotomic divisors of $A$ and $A'$



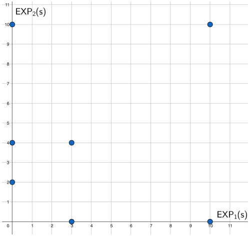



Figure: The cyclotomic divisors of $A$

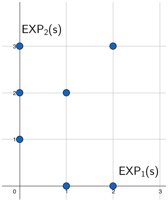



Figure: The cyclotomic divisors of $A'$

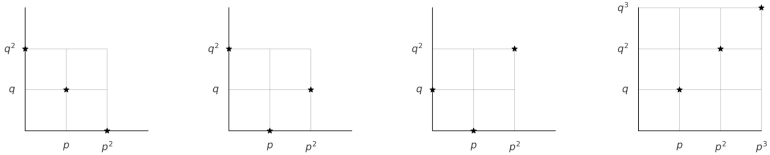
## Theorem

*Let $M = p^{n_1} q^{n_2}$, and let $S = \{s_1, s_2, s_3\}$ and $s_i \mid s_j$ for $i \neq j \in \{1, 2, 3\}$. Then*

$$\mathrm{MIN}(S) \geq (\min(p, q))^3$$

.

**Sketch of proof.** By the truncation procedure and further reduction, we can assume that all of the exponents are at most 2 or we have the diagonal case. So we get the following four cases (up to symmetry):



In our example $M = p^n q^m$ with $n \geq 9$ and $m \geq 6$, and $p = 2, q = 3$

First we define a multiset $B \in \mathcal{M}^+(\mathbb{Z}_{pq})$ $(p = 2, q = 3)$.

| | 0 mod 3 | 1 mod 3 | 2 mod 3 | row sum |
|---|---|---|---|---|
| 0 mod 2 | 74 | 47 | 47 | 21·8 |
| 1 mod 2 | 34 | 7 | 7 | 6·8 |
| column sum | 4·27 | 2·27 | 2·27 | |

Each column sum is divisible by 27 and each row sum by 8.

In our example $M = p^n q^m$ with $n \geq 9$ and $m \geq 6$, and $p = 2, q = 3$

First we define a multiset $B \in \mathcal{M}^+(\mathbb{Z}_{pq})$ $(p = 2, q = 3)$.

| | 0 mod 3 | 1 mod 3 | 2 mod 3 | row sum |
|---|---|---|---|---|
| 0 mod 2 | 74 | 47 | 47 | 21·8 |
| 1 mod 2 | 34 | 7 | 7 | 6·8 |
| column sum | 4·27 | 2·27 | 2·27 | |

Each column sum is divisible by 27 and each row sum by 8.

$B$ is a sum of $p$- and $q$-fibers.

Now we construct a set $A \subset \mathbb{Z}_M$ such that $A \equiv B \mod pq$.

As we have $\Phi_{p^n}\Phi_{p^{n-1}}\Phi_{p^{n-2}} \mid A(X)$, $A \mod p^n$ must be the union of long $p^3$-fibers on scale $M$. $A$ has to be a set, which is guaranteed if they are disjoint.

If $n \geq 9$, then we have enough space for that.

$\Phi_{q^m}\Phi_{q^{m-1}}\Phi_{q^{m-2}} \mid A(X)$ implies that $A \mod q^m$ has to be the union of (disjoint) long $q^3$-fibers.

If $m \geq 6$, then they can be taken to be disjoint.

# $\Phi_p \Phi_{p^2} \Phi_{p^3} \Phi_q \Phi_{q^2} \Phi_M \mid A(X)$ and $|A| = p^3 q^2 = 72$

Now $M = p^4 q^4$, $p = 2, q = 3$.

The following table represents a multiset $B \in \mathcal{M}^+(\mathbb{Z}_{72})$, where the cyclic group $\mathbb{Z}_{72}$ is written as $\mathbb{Z}_8 \oplus \mathbb{Z}_9$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 2 |
| 3 | 4 | 0 | 0 | 0 | 2 | 0 | 0 | 0 |
| 0 | 0 | 5 | 2 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 3 | 2 | 0 | 0 | 0 | 4 | 0 |
| 0 | 0 | 0 | 0 | 5 | 2 | 0 | 0 | 2 |
| 0 | 4 | 0 | 0 | 3 | 2 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 5 | 2 | 2 |
| 0 | 0 | 0 | 4 | 0 | 0 | 3 | 2 | 0 |

The entries in each column add up to 8, and the entries in each row add up to 9. This guarantees that

$$\Phi_p \Phi_{p^2} \Phi_{p^3} \Phi_q \Phi_{q^2} \mid B.$$

Now we construct a set $A \subset \mathbb{Z}_M$ such that $B \equiv A \bmod p^3 q^2$ and $\Phi_M \mid A$.

Each positive entry (2, 3, 4, 5) in the table is a nonnegative integer coefficient linear combination of 2 and 3. Hence, we may define $A$ is each $\mathbb{Z}_{pq^2}$ coset to be either just a single 2-fiber, or a single 3-fiber, or two 2-fibers, or a 3-fiber and a 2-fiber, where each fiber is on scale $M$.

If there is two fibers in one $\mathbb{Z}_{pq^2}$ coset, we place them in different $\mathbb{Z}_{pq}$ cosets of it, guaranteeing that they do not overlap.

Hence $A$ is a set satisfying the requirements.

# Possible generalization



The previous construction can be extended to any primes $p \neq q$.

**Theorem**

*Let $p, q$ be any two distinct primes. We can choose $a, b, n, m \in \mathbb{N}$ ($a << n, b << m$) large enough so that there is a set $A$ of size $|A| = p^a q^b$ that satisfies*

$$\Phi_p \Phi_{p^2} \cdots \Phi_{p^a} \Phi_q \Phi_{q^2} \cdots \Phi_{q^b} \Phi_{p^n q^m} \mid A.$$

# Further generalization



Two natural directions of generalization.

- *simultaneous divisibility* by a block of the form

$$\prod_{L: L_0 | L | M} \Phi_L(X),$$

  where $L_0 = p^\alpha q^\beta \mid M = p^n q^m$ by replacing the single $p$- and $q$-fibers with long $p^{n-\alpha+1}$- and $q^{m-\beta+1}$-fibers.
- The construction can also be extended inductively to the case of *arbitrary finite sets of primes* $\{p_1, \ldots, p_r\}$, provided that the parameters involved are chosen sufficiently large.

# (T2) lower bound



**Theorem**

*Let $M = p^m q^n$ and suppose that $A \in \mathcal{M}^+(\mathbb{Z}_M)$ satisfies $(T2)$. If there exists some $N = p^\gamma q^\eta$ such that $\Phi_N(X) \mid A(X)$ and $\Phi_{p^\gamma}(X), \Phi_{q^\eta}(X) \nmid A(X)$, then*

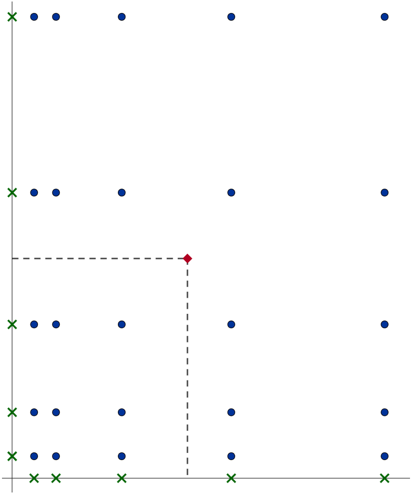
$$A(1) > \prod_{s \in S_A^*} \Phi_s(1), \tag{2}$$

*where $S_A^*$ is the set of prime powers $p^\alpha$ such that $\Phi_{p^\alpha}(X) \mid A(X)$*

In other words, if $A \in \mathcal{M}^+(\mathbb{Z}_{p^m q^n})$ satisfies $(T2)$ and also has an unsupported divisor $\Phi_N(X) \mid A(X)$. Then $A$ has the size increase given in (2).

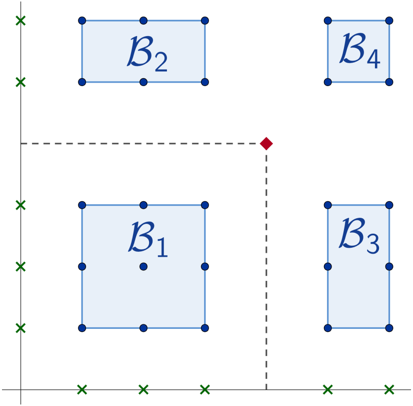
# The original setup





# The truncated version





**Corollary**

*Suppose that* $A \subset \mathbb{N}$ *satisfies (T1) and (T2), and that* $\text{lcm}(S_A) = p^m q^n$ *for two distinct prime factors* $p, q$. *Then* $A$ *does not have any unsupported divisors.*

If $A \oplus B$ is a tile of $\mathbb{Z}_M$ for $M = p^m q^n$, and $A$ satisfies (T2), then $B$ also satisfies (T2).

**Question ('weaker' Coven-Meyerowitz conjecture)**

*Is it true that whenever* $A \oplus B$ *is a tile of a cyclic group and* $A$ *satisfies (T2), then* $B$ *also satisfies (T2)?*

# Negative results in 4D



**Theorem**

*Let* $N = p_1 p_2 p_3 p_4$ *and* $M = N^4$*, where*

$$p_1 > 40 \text{ and } p_i < p_{i+1} < 2p_i \text{ for } i = 1, 2, 3.$$

*Then there exists a set* $A \subset \mathbb{Z}_M$ *such that:*

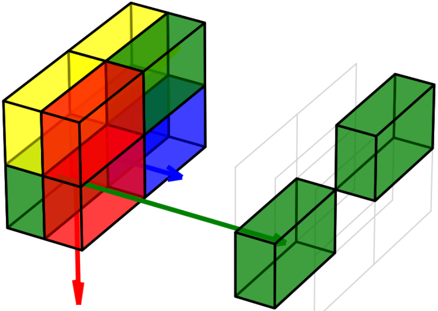
(i) *the prime power cyclotomic divisors of* $A(X)$ *are* $\Phi_{p_i^\alpha}(X)$ *for all* $i = 1, 2, 3, 4$ *and* $\alpha = 2, 3, 4$*,*

(ii) *A satisfies both (T1) and (T2), so that in particular we have* $|A| = N^3$*,*

(iii) *additionally,* $A(X)$ *has the unsupported cyclotomic divisor* $\Phi_N(X)$*.*

# Tricky 4D-fiber shift





# Properties



- Let $A'$ be a $\mathbb{Z}_N^3$ coset, hence $A' = N^3$.
- Originally, we have $\Phi_m \mid A'(X)$ for $m = p_i^k$ ($i \in \{1, 2, 3, 4\}$,$k \in \{2, 3, 4\}$), and all (T2) divisors given by them.
- The shifted set $A$ preserves this divisibility.
- We divide each side such that the shifted part in the $p_i$-direction is divisible by $p_i$, and each $\mathbb{Z}_M/p_i$ coset we shift the same number of $p_i^3$-fibers.
- Thus the set is the sum of $p_i$-fibers, hence $\Phi_N \mid A$.

# Final remarks



- The standard set (which takes one point from each $\mathbb{Z}_{N^3}$-coset) is a tiling complement but it satisfies $(T2)$.
- It can be modified such that $\Phi_s \mid B(X)$ whenever $s|N$ and $s \neq N$, and $\Phi_N \nmid B(X)$ (thus (T2) fails for the set $B$).
- However we cannot guarantee that $A$ tiles with $B$, namely we cannot ensure e.g. that $\Phi_{p_1 p_2 p_3^2 p_4^2} \mid A(X)B(X)$ holds.

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**Question ('weaker' Coven-Meyerowitz conjecture)**

*Is it true that whenever $A \oplus B$ is a tile of a cyclic group and $A$ satisfies (T2), then $B$ also satisfies (T2)?*

Thank you for your kind attention.